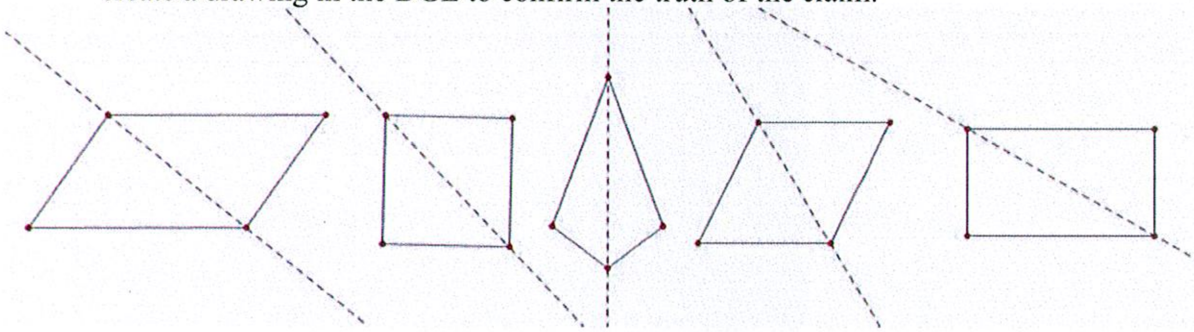
Sample Solutions

Q33. Do you think the two brothers should use this method? Explain why.

No, if the quadrilateral is irregular, the two sections will not necessarily be congruent. So, if the goal of the brothers is to have equal portions of land, this method is not sufficient.

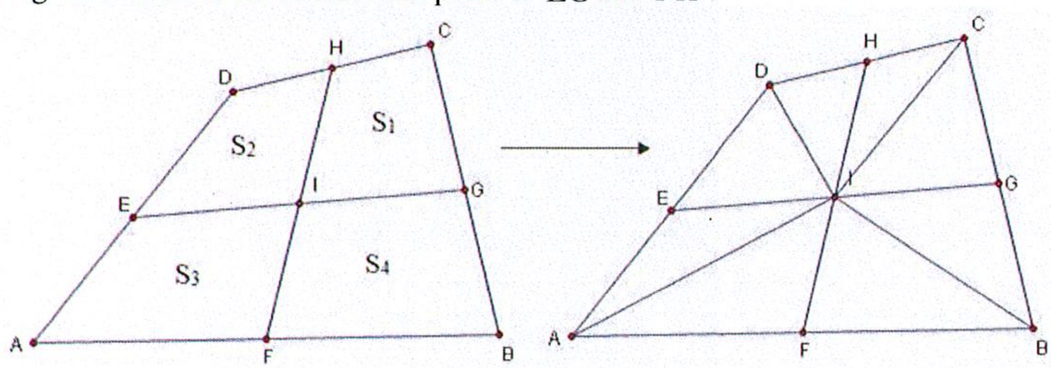
Q34. If the island was of a particular shape, would you suggest using this method? Explain.

For some type of quadrilaterals, this statement would be true. For example, if the island had one of the shapes such as kite, parallelogram, rectangle, rhombus or square, then [at least] one of the diagonals would divide the shape into two equal triangle parts. You can create a drawing in the DGE to confirm the truth of the claim.

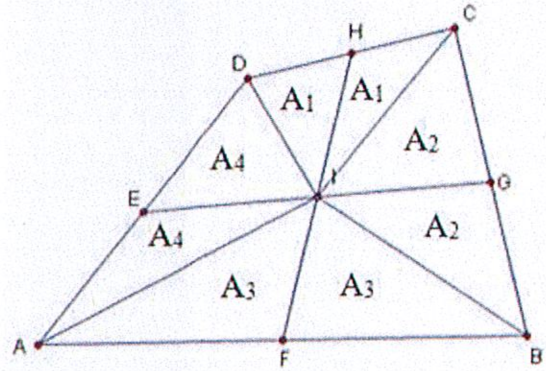


Q35. Use a DGE to determine another method for dividing the land fairly between the two brothers. Be prepared to explain why your method works.

One way to divide the land evenly is to connect the midpoints of each set of opposite sides. This creates four quadrilaterals inside the larger quadrilateral. If each brother takes two of the non-adjacent sections, the total area of their land is equal.
Suppose that ABCD is a quadrilateral and points E, F, G and H are midpoints of the line segments and I is the intersection point of $\overline{EG}$ and $\overline{FH}$.

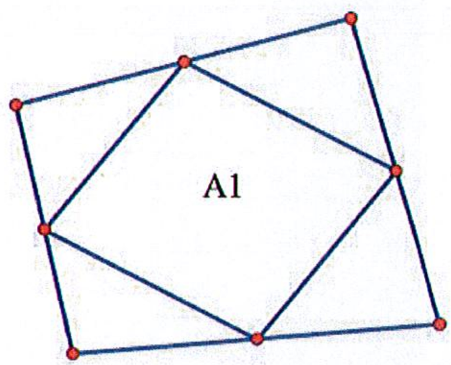


A(DHI) and A(CIH) are equal since their length of bases are same (inasmuch as point H is the midpoint) and they have a the same height. Let's represent each of these two equal areas with A_1. For the same reasons, A(CGI) and A(BGI) [represented by A_2]; A(AFI) and A(BFI) [denoted by A_3]; A(AEI) and A(DEI) [represented by A_4] each pair has equal areas. In this case:



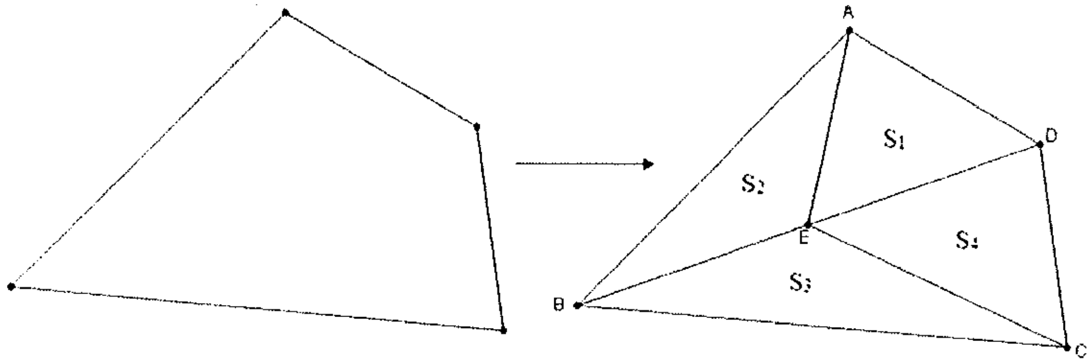
$S_1= A_1+ A_2$; $S_2= A_1+ A_4$; $S_3= A_3+ A_4$ and $S_4= A_2+ A_3$. If we combine these, we get: $S_1+S_3=S_2+S_4$

Another solution involves finding the midpoints of each of the sides of the quadrilateral and then forming the midpoint quadrilateral with segments. There will be two areas that are equal. The area of A1 is equal to the sum of the four triangular regions.



Q36. Suppose the brothers want to make sure they each have land that is on the water. Devise a method that will satisfy each brother. Be prepared to explain your solution.

Suppose that the island had an irregular convex quadrilateral shape as shown below. Then, we could create one of its diagonals [$\overline{BD}$ in the figure] and mark the midpoint of this diagonal (point E in the figure). Since point E is the midpoint of the diagonal, the measurement of $\overline{BE}$ and the measurement of $\overline{DE}$ are the same. For that reason, in the triangle ABD, the area of triangle of AED [S_1] and the area of triangle ABE [S_2] are equal inasmuch as they have both equal heights. Similarly, the area of triangle BCE [S_3] and the area of triangle CDE [S_4] are equal inasmuch as they have both the same heights. If we summarize the equation we see that: $S_1=S_2$ and $S_3= S_4$. In this context, there are two possibilities in order to share the land equally. First, one of the brothers could have $S_1+ S_4$ and the other could have $S_2 +S_3$. Second, one might have S_1+S_3 while the other might have S_2+S_4.



Q37. While considering various methods, the younger brother noticed that if you create the diagonals of the quadrilateral, measure the areas of each of the triangular regions created, and then multiply the area measures of the two non-adjacent triangles, the products are equal. Is this relationship always true? Explain.

Yes, this statement is always true. We can prove it as follows:
Suppose ABCD is a quadrilateral, $\overline{AC}$ and $\overline{BD}$ are the diagonals of the quadrilaterals and point E is the intersection point of the diagonals.

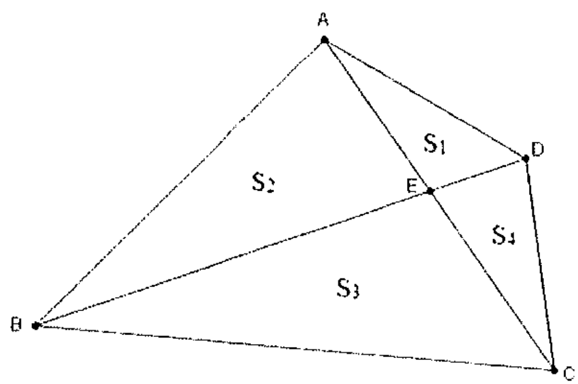
Since triangle ADE [S_1] and triangle ABE [S_2] have the same height; let's denote their heights with h_1. Similarly, triangle BCE [S_3] and triangle CDE [S_4] have the same height; let's denote their heights with h_2.

$$\frac{S_1}{S_2} = \frac{ED.h_1/2}{BE.h_1/2} = \frac{ED}{BE} \quad .....(1)$$

$$\frac{S_4}{S_3} = \frac{ED.h_2/2}{BE.h_2/2} = \frac{ED}{BE} \quad .....(2)$$

Then, from 1 and 2 we get:

$\frac{S_1}{S_2} = \frac{S_4}{S_3}$, which means $S_1.S_3=S_2.S_4$



Q38. Describe the mathematical goals this task addresses.

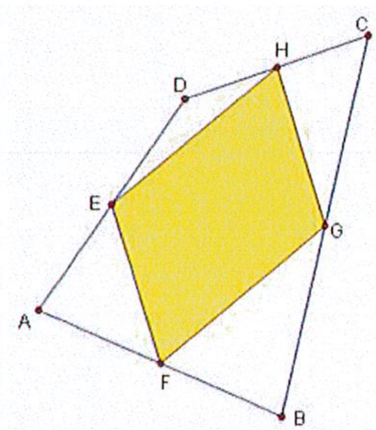
This task requires students to divide a quadrilateral into two equal parts. They are also required to make conclusions about whether or not a diagonal would divide a quadrilateral into equal parts. For that reason, students are expected to reason about the area relationships after dividing the quadrilateral into two or more parts. In this context, in order to reason correctly, students must realize that the areas of two triangles that have the same base are equal if their height is equal, as well.

Q39. If a teacher posed this problem to students in a classroom it is likely that students would develop more than one correct solution. Anticipate the different solutions students might develop. Describe how each of these solutions is related to the mathematical goals.

In any case, students need to divide the quadrilateral into two or more parts. If they divide it into two parts by one of its diagonals and create two separate triangles, inasmuch as the triangles have the same bases, students must only check whether the heights of the triangles are equal. If they prefer to mark the intersection of the diagonals and create four triangles, similarly students must take the equality of the triangles' heights into consideration. If we revise the different solutions we can conclude that students will most likely use the fact that triangles with equal heights will have the same base.

Q40. For each of the solutions you anticipated, which would you have students share with the class. If you would share more than one solution, in what order would you have students present their solutions?

First, it might be best to show a solution where someone created one of the diagonals of the quadrilateral and discuss whether the two separate triangle areas have the same area or not. If not, the solution that is given at A.36 might be useful to show next, because by observing the height of the triangles, students will have an opportunity to reason that triangles that have the same base have equal area. Another advantage of this solution is that it does not require mathematical computation or a formal proof. The next preferred solution might be using the solution given in A35. Although this solution shows a lot of separate parts, the key idea of this solution is similar to the last one: using triangles that have equal bases and the same height to show their areas are equivalent. One can also use Varignon's theorem in order to create two equal areas. This theorem says "the figure formed when the midpoints of the sides of a convex quadrilateral are joined in order is a parallelogram. Equivalently, the bimedians bisect each other. The area of the Varignon parallelogram of a convex quadrilateral is half that of the quadrilateral…" (Retrieved from http://www.mathopenref.com/square.htmlhttp://mathworld.wolfram.com/VarignonsTheorem.html, April 30, 2012).



Varignon Theorem: ABCD is a quadrilateral; points E, F, G and H are the midpoints of each line segment. Then, EFGH is a parallelogram and A(EFGH)=A(ABCD)/2

On the other hand, the solution that is given in A37 is an interesting property about areas of triangles that are created by the diagonals of a quadrilateral. However, it is not typical to use this property to create equal areas.

Q41. What do you notice about Triangle EFG? Create at least two different interesting conjectures based on the diagram.

The triangle formed by the midpoints of the diagonals and the midpoint of one non-congruent sides is always an isosceles triangle. Also, each of the two congruent legs of the isosceles triangle is equal to half the length of the either congruent side of the quadrilateral.

Q42. Select one of the two conjectures and determine whether it is never true, sometimes true, or always true.

One of the congruent legs of the isosceles triangle is always equal to half the length of one of the congruent sides of the quadrilateral. Similarly, the triangle formed by the midpoints of the diagonals and the midpoint of one of the non-congruent sides is always isosceles.

Q43. To determine whether a conjecture is always true it is likely that you constructed a mathematical argument or proof. Describe how your work with the technology was related to the construction of your argument.

Students may or may not construct an actual mathematical argument or proof. Rather, they are likely to visually explore relationships through dragging and even measuring parts of the figure(s).

Q44. Describe some ways that a teacher can assist students in using the technology to gather data that can assist them in constructing formal mathematical arguments.

If a teacher notices that students are having trouble crafting a mathematical argument and are making guesses haphazardly, he/she can remind students to manipulate the figures within the DGE. Encouraging students to continue to explore with technology and discussing their ideas with classmates is a critical part of the teacher's role in this type of activity. The teacher may also suggest students to consider what remains the same as they are dragging. These invariances are often related to properties that can be used in a formal argument.

Q45. Create a new problem that extends the given situation and provide a sample solution. Recall that problem-posing strategies were presented in Chapter 2.

There are many possibilities. One could be: Consider what happens if we construct a triangle using the midpoints of the diagonals and the midpoint of one of the other non-congruent sides. Are the results the same? Explain.

Q46. Consider the quadrilateral problem presented at the beginning of Section 5. Describe the dragging strategies that you used and the purpose for using that particular type of dragging.

Since point E, F and G are midpoints of the line segments they will only translate the shape holistically if dragged. However, the independent points, the vertices of the quadrilateral, will allow you to manipulate the shape, which provides an opportunity to observe the mathematical properties of the shape. Some students may initially engage in wandering dragging to see what happens to the figure when a vertex is moved, then, they may move into guided dragging if they hypothesize making a certain shape can confirm their conjectures. Once students feel confident in their conclusions, he/she may engage in a dragging test or dummy locus dragging to further confirm conjectures.

Q47. Were certain dragging strategies more or less useful in solving the problem? Explain.

For this particular problem, it seems that wandering dragging is helpful initially, but once one realizes the relationships, dragging tests become more helpful in making sure those relationships are maintained regardless of how a vertex is moved.

Q48. Would you want to present these different dragging methods to students? Why or why not?

It would be helpful to present these different dragging methods to students so they can better understand how they are engaging with the technology. By understanding these methods, students would be better equipped to describe the methods used to arrive at a particular answer.

Q49. View the videoclip, "Dragging.mov." Describe how observing the ways in which this student is dragging provides insights into what he or she is thinking.

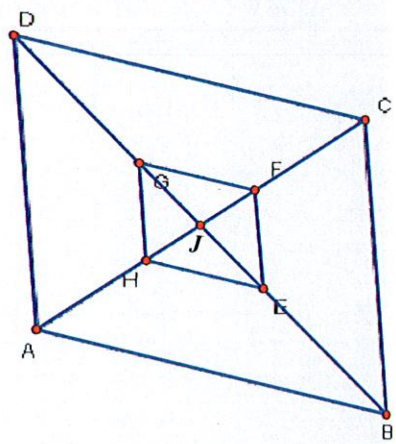
Q50. What type of quadrilateral is EFGH? How did you arrive at your conclusion?

Quadrilateral EFGH is also a parallelogram. For example, the DGE may be used to measure the slope of each of the line segments. Using the dragging techniques mentioned earlier, students could verify that opposite sides of EFGH are always parallel.

Q51. Determine the "given(s)" and the statement "to prove" and construct a formal argument to validate the claim you made in Q47.

Given: ABCD is parallelogram. E, F, G, and H are the centroids of triangles ABC, BCD, CDA, and DAB.

Prove: Quadrilateral EFGH is a parallelogram.



1. ABCD is parallelogram. E, F, G, and H are the centroids of triangles ABC, BCD, CDA, and DAB.	1. Given
2. Let J be the intersection of the two diagonals of parallelogram ABCD	2. If two lines intersect, then they intersect in a single point.
3. JD=JB, JA=JC	3. If a quadrilateral is a parallelogram, then the diagonals bisect each other.
4. JG=1/3JD, JE=1/3JB, JH=1/3JA, JF=1/3JC	4. If the centroid of a triangle is on a median of a triangle, then the centroid divides the median at a ratio of 2:1.
5. 1/3JD=1/3JB, 1/3JA=1/3JC	5. If both sides of an equation are multiplied by the same number, then the results are equal
6. JG=JE, JH=JF	6. Substitution Axiom
7. Quadrilateral EFGH is a parallelogram.	7. If the diagonals of a quadrilateral bisect each other, then the quadrilateral is a parallelogram.

Q52. What prerequisite knowledge would students need to know in order to construct a proof such as the one that you created in Q48?

Students would need to know the properties of both parallelograms and centroids of a triangle and the theorems associated with each. Students would also need to know the properties of equality and understand that we can use constructions (like step 2) to aid in proofs.

Q53. Create four additional tasks that you could pose to students that are extensions or modifications of this problem.

Projects could include: 1) Compare the areas of the two parallelograms 2) What sets of parallel lines do you see in this figure? Why? 3) What is the relationship between triangle AJD and triangle HJD? Explain. 4) Is parallelogram ABCD similar to parallelogram HEFG? Why or Why not?